**Tensors and stuff**

(note we’re using Einstein summation convention, implicitly; maybe see Appendix at bottom)

**6. Special Differential Vector Operators**

Now we want to look at the general features of differentiating scalars, vectors and tensors. And in particular we’ll consider how to generally write vector operators like the gradient, divergence, curl, and Laplacian.

**6a. Gradient**

Generally speaking, the gradient is defined as:



It is important to note that this definition is a true vector, meaning that the components ∂/∂ui

do transform covariantly. It must be the case that the gradient, divergence, curl, and Laplacian transform as vectors, scalars, vectors, and scalars respectively if they are to have a geometric meaning beyond just a single coordinate system. Suppose we take the gradient of a vector:



Note this must be a tensor, because **T** is a tensor, and taking the gradient of a tensor is a tensor. What is the interpretation of this construct? I think it’s something like this:



just like ∇φ is the **i·**rate of change of φ in x direction + **j**·rate of change of φ in y direction, etc. Or similarly, maybe we can think of (∇T)αβ as being the rate of change of the Tα component in the β direction. Anyway, let’s work this out.



where we define the covariant derivative of T w/r to variable i as:



and so we can then say:



The outer product of two vectors must give a tensor, right? So is it true that T´α;β transform as the components of a tensor?



So it checks out. Though, I had to use the inverted form of the Christoffel symbol transformation law. We can expand this to cover higher order tensors. For instance,



So the covariant derivative in this case is:



By the same reasoning we have:



and,



Observe how the free indices keep the same positions in all cases, and there is a minus sign when its on the lower index. Does the integral version of this hold? Say we’re operating on a scalar.



Seems pretty clear it does since the integrand is simply a total differential. What if we operate on a tensor instead?



Well my only reservation would be that we’re subtracting a tensor, defined w/r to one basis set, from one defined w/r to another basis set. And I’m not sure this is possible per se´? So not sure. As we’ll see later, a manifold with curvature has non-commuting cross-partials. So that would seem to also put this result in doubt b/c it would suggest that going from a to b via different paths would not be equivalent.

**Example**

What is the gradient in spherical coordinates? Well we previously determined that:



but these aren’t the usual spherical coordinate unit vectors, which are normalized. So let’s determine the normalization of these guys.



So in terms of unit vectors we have:



and so the gradient follows as:



**6b. Divergence**

The divergence is defined as:



of course. Let’s operate on a two-component tensor. Then,



So we have, for instance,



If we were to just operate on a one component tensor then we’d have:



Explicitly, we have:



If the derivatives of the basis vectors were zero, then only the first term would survive. And this is the expression for the divergence in the Cartesian coordinate system where  don’t change with position. However in cylindrical or spherical coordinates, etc., the basis vectors do and so we have to examine the contribution from the first term. Of course we have just examined this contraction of Γ. And so we have:



So we have:



Integrating this, we obtain Gauss’s law:



**Example**

What is the divergence in spherical coordinates? Well knowing that:



so that:



we have:



However, the components fr, fθ, and fφ are here defined with respect to the un-normalized basis vectors **e­**r, **e**θ, and **e**φ. If we redefine the components to be w/r to the normalized basis vectors. Then this would read:



so in spherical coordinates:



which is the correct expression.

**6c. Curl**

Now we’ll generalize the curl to possibly position-dependent basis vectors. The curl of a vector field is defined as:



just as before. Working this out.



and this eventually simplifies to:



**6d. Laplacian**

The Laplacian is easy to write down. It is simply the divergence of the gradient and so:



Using our expression for the divergence and gradient we have:



So we have:



**Example**

What is the Laplacian in spherical coordinates? Well we have that:



and,



and so,



Filling these in we get:



So in spherical coordinates:



which is the known expression. As you can see, this is a lot easier than doing the change of variables directly on the Cartesian expression for the Laplacian. Plus this method already puts it in the simplified differential form, aiding in the solution of the equation itself. So for instance, if you wanted to calculate the Laplacian in parabolic coordinates (another useful coordinate system) then you would probably want to do it like this.